QUANTIZED TIMELINES

JACK MORAVA

Abstract. Conformal nets are a classical [5] topic in quantum field theory: they assign operator algebras to one-dimensional manifolds, and have close connections with one-dimensional topological field theories [1,4,7].

It seems to be well-known that the usual axioms for these constructions imply close relations between the action of the projective group on the line, and Connes' intrinsic flow on C^*-algebras. This note attempts to pin down this specific fact, in terms of a category of (noncommutative) algebras and equivalence classes, under inner automorphisms, of homomorphisms between them. That category may be of independent interest.

1. A two-category of algebras

1.0 In the following, $A, B, C \ldots$ will be k-algebras, eg for $k = \mathbb{C}$, and $\mathrm{Hom}(A, B)$ will denote the set of algebra homomorphisms between them.

$$\sigma_a(x) = a^{-1}xa := x^a$$

will denote conjugation by a unit a in some algebra (so $\sigma_{ab} = \sigma_b \circ \sigma_a$). In fact the algebras of most interest here will be topological, and the homomorphisms will usually be continuous, but I'll leave that in the background.

Let $\mathbf{H}\mathrm{om}(A, B)$ be the category with $\mathrm{Hom}(A, B)$ as its set of objects, and morphisms $(a, b) : \phi_0 \to \phi_1$ defined by

$$\mathrm{mor}(\phi_0, \phi_1) := \{(a, b) \in A^\times \times B^\times \mid \sigma_b \circ \phi_0 = \phi_1 \circ \sigma_a\}$$

ie $\phi_1(x) \cdot \phi_1(a)b^{-1} = \phi_1(a)b^{-1} \cdot \phi_0(x)$ or, alternately,

$$\sigma_{\phi_1(a)b^{-1}}(\phi_1(x)) = \phi_0(x) \ .$$

Note, for future reference, that $\phi_1(a)b^{-1} = b^{-1}\phi_0(a)$: take $x = a$ in the defining condition for a morphism.

Date: 21 May 2012.
1991 *Mathematics Subject Classification.* 16Wxx, 46Lxx, 81Sxx.

Morphisms $(a_0, b_0) \in \mathrm{mor}(\phi_0, \phi_1)$ and $(a_1, b_1) \in \mathrm{mor}(\phi_1, \phi_2)$ compose according to the diagram

$$\begin{tikzcd} A \arrow[r, "a_0"] \arrow[d, "\phi_0"] & A \arrow[r, "a_1"] \arrow[d, "\phi_1"] & A \arrow[d, "\phi_2"] \\ B \arrow[r, "b_0"] & B \arrow[r, "b_1"] & B \end{tikzcd}$$

ie

$$(a_1, b_1) \circ (a_0, b_0) := (a_0 a_1, b_0 b_1) \in \mathrm{mor}(\phi_0, \phi_2) .$$

$\mathbf{H}\mathrm{om}(A, B)$ is thus a groupoid. For example, the group of automorphisms of ϕ consists of pairs $a \in A^\times, b \in B^\times$ such that $\phi(a)b^{-1}$ lies in the center of the image of ϕ: thus

$$\phi(\alpha)\beta^{-1} \cdot \phi(a)b^{-1} = \phi(a)\phi(\alpha)\beta^{-1}b^{-1} = \phi(a\alpha)(b\beta)^{-1} .$$

1.1 Proposition: The composition law

$$(\phi, \psi) \mapsto \psi \circ \phi : \mathbf{H}\mathrm{om}(A, B) \times \mathbf{H}\mathrm{om}(B, C) \to \mathbf{H}\mathrm{om}(A, C)$$

defined on morphisms by

$$(a, b_0) \times (b_1, c) \mapsto (a, c\psi_1(b_1^{-1}b_0)) : \psi_0 \circ \phi_0 \mapsto \psi_1 \circ \phi_1$$

defines a two-category $\mathbf{A}\mathrm{lg}_{\mathrm{Out}}$ (of algebras and homomorphisms, up to inner automorphism).

Proof: This composition law is well-defined on morphisms: we have

$$\phi_1(a)b_0^{-1} \cdot \phi_0(x) = \phi_1(x) \cdot \phi_1(a)b_0^{-1}$$

and

$$\psi_1(b_1)c^{-1} \cdot \psi_0(y) = \psi_1(y) \cdot \psi_1(b_1)c^{-1}$$

$(\forall x, y \in A, B,$ with $a \in A,\ b_0, b_1 \in B, c \in C)$, so if we apply ψ_0 to the first expression, and left-multiply both sides by $\psi_1(b_1)c^{-1}$, we get

$$\psi_1(b_1)c^{-1} \cdot \psi_0(\phi_1(a)b_0^{-1}) \cdot (\psi_0\phi_0)(x) = \psi_1(b_1)c^{-1} \cdot \psi_0(\phi_1(x)) \cdot \psi_0(\phi_1(a)b_0^{-1}) ,$$

which in turn equals

$$\psi_1(\phi_1(x)) \cdot \psi_1(b_1)c^{-1} \cdot \psi_0(\phi_1(a)b_0^{-1}) ;$$

that is, conjugation by $\psi_1(b_1)c^{-1}\psi_0(\phi_1(b_1)b_0^{-1})$ sends $\psi_1(\phi_1(x))$ to $\psi_0(\phi_0(x))$.

On the other hand, it follows from the second relation above that

$$\psi_1(b_1)c^{-1} \cdot \psi_0(\phi_1(b_1)b_0^{-1}) = \psi_1(\phi_1(a)b_0^{-1}) \cdot \psi_1(b_1)c^{-1} = \psi_1(\phi_1(a)b_0^{-1}b_1)c^{-1} ,$$

so

$$\sigma_{(\psi_1 \circ \phi_1)(a)(c\psi_1(b_1^{-1}b_0))^{-1}}((\psi_1 \circ \phi_1)(x)) = (\psi_0 \circ \phi_0)(x)$$

ie $(a, c\psi_1(b_1^{-1}b_0)$ defines a morphism from $\psi_0 \circ \phi_0$ to $\psi_1 \circ \phi_1$.

It remains to check associativity. Let $\phi, \psi, (a, b_0), (b_1, c)$ be as above, and let

$$(z, \tilde{a}) \in \mathrm{mor}(\theta_0, \theta_1)$$

for morphisms $\theta_0, \theta_1 \in \mathrm{Hom}(Z, A)$; then

$$(z, \tilde{a}) \times (a, b_0) = (z, b_0\phi_1(a^{-1}\tilde{a})) : \theta_0 \times \phi_0 \to \theta_1 \times \phi_1 \ ,$$

so

$$((z, \tilde{a}) \times (a, b_0)) \times (b_1, c) = (z, c\psi_1(b_1^{-1}b_0\phi_1(a^{-1}\tilde{a}))) : (\theta_0 \times \phi_0) \times \psi_0 \to (\theta_1 \times \phi_1) \times \psi_1 \ ;$$

while

$$(a, b) \times (b_1, c) \mapsto (a, c\psi_1(b_1^{-1}b_0)) : \phi_0 \times \psi_0 \to \phi_1 \times \psi_1 \ ,$$

so

$$(z, , \tilde{a}) \times ((a, b)) \times (b_1, c)) = (z, c\psi_1(b_1^{-1}b_0)(\psi_1\phi_1)(a^{-1}\tilde{a})) \ .$$

□

1.2 Definition By analogy with the construction of the group $\mathrm{Out}(G) = \mathrm{Aut}(G)/(G/Z(G))$ of outer automorphisms of a group G, let

$$\mathrm{Hom}_{\mathrm{Out}}(A, B) := \pi_0\mathbf{H}\mathrm{om}(A, B)$$

be the set of isomorphism classes of objects in the groupoid $\mathbf{H}\mathrm{om}(A, B)$; this defines a category $\mathrm{Alg}_{\mathrm{Out}}$ whose objects are algebras, and whose morphisms are equivalence classes, up to inner automorphism, of algebra homomorphisms.

Note that a more familiar two-category of associative algebras takes bimodules as its morphisms.

2. A two-category of one-manifolds

2.1 Let $I, J, \ldots$ be compact connected oriented Riemannian one-manifolds (roughly: 'intervals'), and let $\mathbf{E}\mathrm{mb}(I, J)$ be the category with oriented smooth embeddings

$$\epsilon : I \to J \in \mathrm{Emb}(I, J)$$

as objects, and diagrams

$$\begin{tikzcd} I \arrow[r, "\epsilon_0"] \arrow[d, "a"] & J \arrow[d, "b"] \\ I \arrow[r, "\epsilon_1"] & J \end{tikzcd}$$

as morphisms $(a, b) : \epsilon_0 \to \epsilon_1$: where ϵ_0, ϵ_1 are embeddings as above, and a, b are (orientation-preserving) diffeomorphisms supported in the **interior** of the interval (ie each equals the identity in a neighborhood of the boundary of its domain).

2.2 Lemma: If $\epsilon \in \mathrm{Emb}(I,J)$ and $c \in \mathrm{Diff}_0(I)$ is a diffeomorphism of I supported, as above, in the interior of I, then

$$c^\epsilon(t) := (\epsilon \circ c)(\epsilon^{-1}(t)) \text{ if } t \in \text{image } \epsilon$$

(and $= t$ otherwise) defines an element of $\mathrm{Diff}_0(J)$ such that

$$\begin{tikzcd} I \arrow[r, "\epsilon"] \arrow[d, "c"] & J \arrow[d, dotted, "c^\epsilon"] \\ I \arrow[r, "\epsilon"] & J \end{tikzcd}$$

commutes. □

2.3 Proposition: The composition law

$$(\epsilon, \delta) \mapsto \delta \circ \epsilon : \mathbf{E}\mathrm{mb}(I,J) \times \mathbf{E}\mathrm{mb}(J,K) \to \mathbf{E}\mathrm{mb}(I,K)$$

defined on morphisms by

$$(a, b_0) \times (b_1, c) \mapsto (a, c(b_1^{-1}b_0)^{\delta_0})$$

defines a two-category $\mathfrak{I}_\partial$ (of intervals with collared boundaries).

Proof: The diagram

$$\begin{tikzcd} I \arrow[r, "\delta_0\epsilon_0"] \arrow[d, "a"] & K \arrow[d, "c(b_1^{-1}b_0)^{\delta_0}"] \\ I \arrow[r, "\delta_1\epsilon_1"] & K \end{tikzcd}$$

commutes, since

$$c\delta_0 b_1^{-1} b_0 \delta_0^{-1} \cdot \delta_0\epsilon_0 = c\delta_0 b_1^{-1} \cdot \epsilon_1 a = \delta_1 b_1 \cdot b_1^{-1}\epsilon_1 a = \delta_1\epsilon_1 a \; .$$

To check associativity, let $(a, b_0) : \epsilon_0 \to \epsilon_1 \in \mathbf{E}\mathrm{mb}(I,J)$, $(b_1, c_0) : \delta_0 \to \delta_1 \in \mathbf{E}\mathrm{mb}(J,K)$, and $(c_1, d) : \eta_0 \to \eta_1 \in \mathbf{E}\mathrm{mb}(K,L)$; then

$$(a,b_0),((b_1,c_0),(c_1,d)) \mapsto ((a,b_0),(b_1,d(c_1^{-1}c_0)^{\eta_0})) \mapsto (a, s(c_1^{-1}c_0)^{\eta_0}(b_1^{-1}b_0)^{\eta_0\delta_0}) \; ,$$

while

$$((a,b_0),(b_1,c_0)),(c_1,d) \mapsto ((a, c_0(b_1^{-1}b_0)^{\delta_0}),(c_1,d)) \mapsto (a, d(c_1^{-1}c_0(b_1^{-1}b_0)^{\delta_0})^{\eta_0}) \; .$$

□

2.4 Definition As the categories $\mathbf{E}$mb are groupoids, we can define a category $\mathfrak{I}_\partial$ with intervals $I, J, \dots$ as objects, and

$$\mathrm{Mor}_{\mathfrak{I}}(I,J) = \pi_0 \mathbf{E}\mathrm{mb}(I,J) \; ;$$

roughly speaking, it is a category of one-manifolds with boundary conditions. Note that the automorphism group of I in $\mathfrak{I}$ is the quotient group $\mathrm{Diff}(I)/\mathrm{Diff}_0(I)$.

3. Conformal nets

3.1 Following Bartels, Douglas, and Henriques [[1,4]; see also [2,5,6]], a conformal net is a kind of cosheaf of von Neumann algebras on a category of suitably oriented compact one-dimensional Riemannian manifolds and smooth embeddings. This note is concerned with a weaker notion of a (continuous) functor $\mathfrak{A}$ on such a category, taking values in a category of topologized algebras, with one bit of extra structure: suppose that

- to any diffeomorphism a of I which is supported on the interior (ie, which leaves a neighborhood of ∂I pointwise invariant), there is an element $\mathfrak{a}(a) \in \mathfrak{A}(I)$ such that $\forall x \in \mathfrak{A}(I)$,

$$\mathfrak{A}(a)(x) = \sigma_{\mathfrak{a}(a)}(x) ,$$

and

- if a_0, a_1 are two such interior automorphisms of I, then

$$\mathfrak{a}(a_0 \circ a_1) = \mathfrak{a}(a_1) \circ \mathfrak{a}(a_0) .$$

(The order reversal is intentional).

3.2 Definition I'll call such a functor a weak quantization for one-manifolds. The condition above is stronger than the analogous (sixth) condition considered in [1 §3.7], but it is satisfied in the example below.

Proposition: Such a weak quantization defines a (strict) two-functor

$$\mathfrak{A} : \boldsymbol{\mathfrak{I}}_\partial \to \mathbf{A}\mathrm{lg}_{\mathrm{Out}}$$

which descends to a plain vanilla functor

$$\mathfrak{A}_0 : \mathfrak{I}_\partial \to \mathrm{Alg}_{\mathrm{Out}} .$$

of ordinary categories.

Proof: To simplify notation, let $\mathfrak{A}(\delta) = D$; and if $a, b, \dots$ are diffeomorphisms supported on the interior of their domains, let $\mathfrak{a}(a) = \mathbf{a}$, $\mathfrak{a}(b) = \mathbf{b}$, etc.

The assertion amounts to checking the commutativity of the diagram

$$\begin{tikzcd}
\mathbf{E}\mathrm{mb}(I,J) \times \mathbf{E}\mathrm{mb}(J,K) \arrow[r] \arrow[d] & \mathbf{E}\mathrm{mb}(I,K) \arrow[d] \\
\mathbf{H}\mathrm{om}(\mathfrak{A}(I),\mathfrak{A}(J)) \times \mathbf{H}\mathrm{om}(\mathfrak{A}(J),\mathfrak{A}(K)) \arrow[r] & \mathbf{H}\mathrm{om}(\mathfrak{A}(I),\mathfrak{A}(K))
\end{tikzcd} ,$$

ie (using the notation above) that

$$\begin{tikzcd} (a,b_0),(b_1,c) \arrow[r] \arrow[d] & (a,c(b_1^{-1}b_0)^{\delta_0}) \arrow[d] \\ (\mathbf{a},\mathbf{b}_0),(\mathbf{b}_1,\mathbf{c}) \arrow[r] & \mathbf{c}D_1(\mathbf{b}_1^{-1}\mathbf{b}_0) \end{tikzcd}$$

commutes: in other words, that

$$\mathfrak{a}(c(b_1^{-1}b_0)^{\delta_0}) = D_0(\mathbf{b}_1^{-1}\mathbf{b}_0)\mathbf{c}$$

(using the fact, from §1.0, that $D_1(\mathbf{b}_1^{-1}\mathbf{b}_0)\mathbf{c}^{-1} = \mathbf{c}^{-1}D_0(\mathbf{b}_1^{-1}\mathbf{b}_0))$. $\square$

3.3 The group $\mathrm{PGl}_2(\mathbb{R})$ acts on the real projective line $P_1(\mathbb{R})$, and the noncompact torus

$$t \mapsto \left[\begin{array}{cc} \cosh t & \sinh t \\ \sinh t & \cosh t \end{array} \right]$$

of Lorentz rotations preserves the interval $\mathbf{I} = [-1,+1]$, defining a homomorphism

$$\mathbb{R} \to \mathrm{Diff}(\mathbf{I})/\mathrm{Diff}_0(\mathbf{I}) \; .$$

[I am indebted to André Henriques for pointing out the interest (and accessibility!) of this quotient.]

A weak quantization $\mathfrak{A}$ for one-manifolds, in the sense above, thus defines a homomorphism

$$\mathbb{R} \to \mathrm{Out}(\mathfrak{A}(\mathbf{I})) \; .$$

On the other hand, Connes [3] exhibits a **canonical** homomorphism

$$\mathbb{R} \to \mathrm{Out}(\mathfrak{A})$$

to the group of outer automorphisms of **any** von Neumann algebra $\mathfrak{A}$ (and has suggested that it be regarded as the flow defined by a kind of intrinsic time).

Wassermann, for example [8 §15] has shown that the free fermion functor $\mathfrak{F}$ (defined by the Fock representation of the Clifford algebra on the L^2 functions on a metrized interval, cf. also [1 §4.1, 7 §4.3.5]) assigns a type III von Neumann algebra to a compact interval, and that it defines a conformal net for which the diagram

$$\begin{tikzcd} \mathbb{R} \arrow[r, "="] \arrow[d] & \mathbb{R} \arrow[d] \\ \mathrm{Diff}(I)/\mathrm{Diff}_0(I) \arrow[r] & \mathrm{Out}(\mathfrak{F}(I)) \end{tikzcd}$$

commutes.

3.4 This seems to me an intriguing fact: it asserts that for the free fermion functor, Connes' intrinsic flow agrees with the natural flow defined by the geometry of the projective line, thus providing one of the few examples in mathematics in which Lorentz geometry appears naturally.

Acknowledgements This work was supported by the NSF; it grew out of discussions at a June 2009 AIM workshop in Palo Alto. I'd like to thank Hisham Sati for organizing that meeting, and its participants for their interest. André Henriques and Chris Douglas get special thanks for generously sharing their time and ideas.

References

1. A. Bartels, C. Douglas, A. Henriques, Conformal nets and local field theory, `arXiv:0912.5307`
2. S. Carpi, Y. Kawahigashi, R. Longo, Structure and classification of superconformal nets, `arXiv:0705.3609`
3. A. Connes, **Noncommutative geometry**, Academic Press (1994); cf also `http://en.wikipedia.org/wiki/Tomita-Takesaki_ theory`
4. C. Douglas, A. Henriques, Topological modular forms and conformal nets, `arXiv:1103.4187`
5. F. Gabbiani, J. Fröhlich, Operator algebras and conformal field theory. Comm. Math. Phys. 155 (1993) 569–640;
6. D. Manolopoulos, I. Runkel, A monoidal category for perturbed defects in conformal field theory, `arXiv:0904.1122`
7. S. Stolz, P. Teichner, What is an elliptic object? in **Topology, geometry and quantum field theory**, 247–343, LMS Lecture Notes 308, CUP (2004)
8. A. Wassermann, Operator algebras and conformal field theory III. Fusion of positive energy representations of LSU(N) using bounded operators. Invent. Math. 133 (1998) 467–538.

The Johns Hopkins University, Baltimore, Maryland 21218
E-mail address: `jack@math.jhu.edu`